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B.Sc. Part-II , Paper IV

Differential Equation (Singular Solution) → Contd.

**Envelope**: Let $f(x, y, c) = 0$ be any given family of curves where $c$ is a parameter.
Consider any two members

$$f(x, y, c) = 0 \quad \text{——— (1)}$$

and $$f(x, y, c+\delta c) = 0 \quad \text{——— (2)}$$

corresponding to the parameters $c$ and $c+\delta c$.
The points common to the two curves (1) and (2) satisfy the equation

$$f(x, y, c+\delta c) - f(x, y, c) = 0$$

$$\Rightarrow \quad \frac{f(x, y, c+\delta c) - f(x, y, c)}{\delta c} = 0$$

Let $\delta c \to 0$. Therefore the limiting position of the points common to (1) and (2) satisfy the eqn.

$$f_c(x, y, c) = 0 \quad \text{——— (3)}$$

**Defn**:- The envelope of family of curves $f(x, y, c) = 0$ is the locus of the limiting position of the points of intersection of the neighbouring curves as $c$ varies continuously.
Thus the coordinates of the points on the envelope i.e. characteristic points satisfy the equation (1) and (3), unless they satisfy also $f_x = f_y = 0$ when they become singular points.

**Defn:-** A point $P(a,b)$ is a singular point of a curve $f(x,y,c)=0$ ($c$ is fixed) if it satisfies besides the equation of the curve, the two relations $f_x=0$, $f_y=0$.

The point $p$ is said to be an ordinary point if at least one of the two partial derivatives $f_x$, $f_y$ is not zero at $(a,b)$.

**Rule:-** To obtain the envelope of the family of curves $f(x,y,c)=0$, if there exists an envelope

1) Eliminate $c$ between $f(x,y,c)=0$ and $\frac{\partial f}{\partial c} f(x,y,c)=0$ — (A)

The c-eliminate (an expression in $x$ and $y$) is the envelope

2) Solve for $x$ and $y$ in terms of $c$ from the equation (A). It will give the parametric representation of the envelope

The equation, obtained by eliminating $c$ between $f=0$ and $f_c=0$ is exactly the condition that the relation $f(x,y,c)=0$ considered as an equation in $c$ has a repeated root. This is particularly simple when $f(x,y,c)=0$ is a quadratic equation in $c$. Thus if $f(x,y,c)=A(x,y)c^2+B(x,y)c+C(x,y)=0$ is a quadratic equation in $c$, then its c-discriminant

$$=B^2-4AC=0.$$

Thus, for example, we proceed to obtain envelope of the family of lines $y=mx+\frac{a}{m}$; $m$ is a parameter

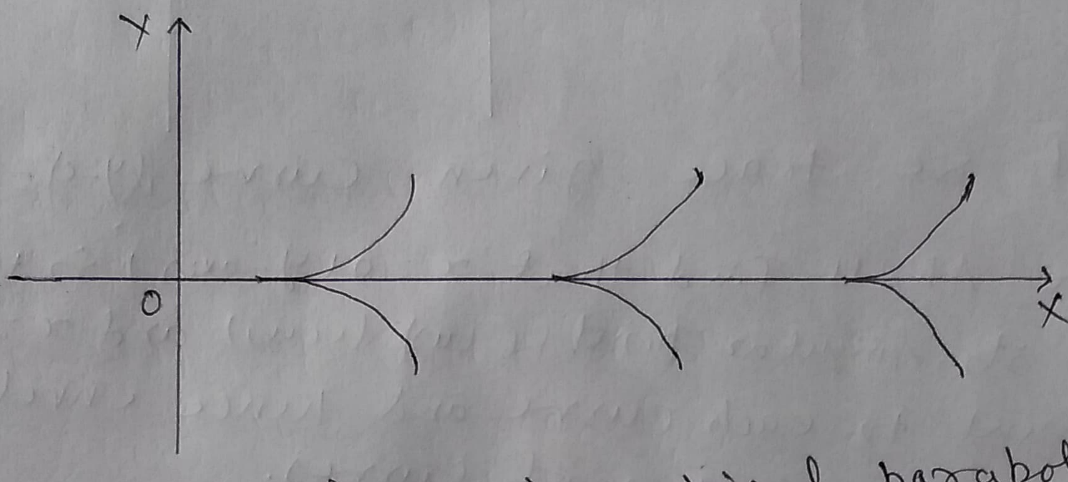
i.e. $m^2x-my+a=0$; a quadratic in $m$.

Then the condition for a double root is $y^2-4ax=0$ (Parabola). Which gives the required envelope

1. Example - Consider the family of Semi-cubical parabolas $f(x,y,c) = y^2 - (x+c)^3 = 0$.

Soln. Here $\frac{\partial f}{\partial c} = 3(x+c)^2 = 0$.

Eliminating c, we obtain $y^2 = 0$ i.e. $y = 0$. It can be verified that $(-c, 0)$ is a singular point (a cusp). Thus $y = 0$ is the locus of singular points, not an envelope, even though it is c-discriminant. The locus here is called cusp locus.




Family of Semi-cubical parabolas.

[Here the neighbouring curves donot intersect each other]

2. Example - Consider the family of curves

$x^2(x-a) + (x+a)(y-c)^2 = 0$.

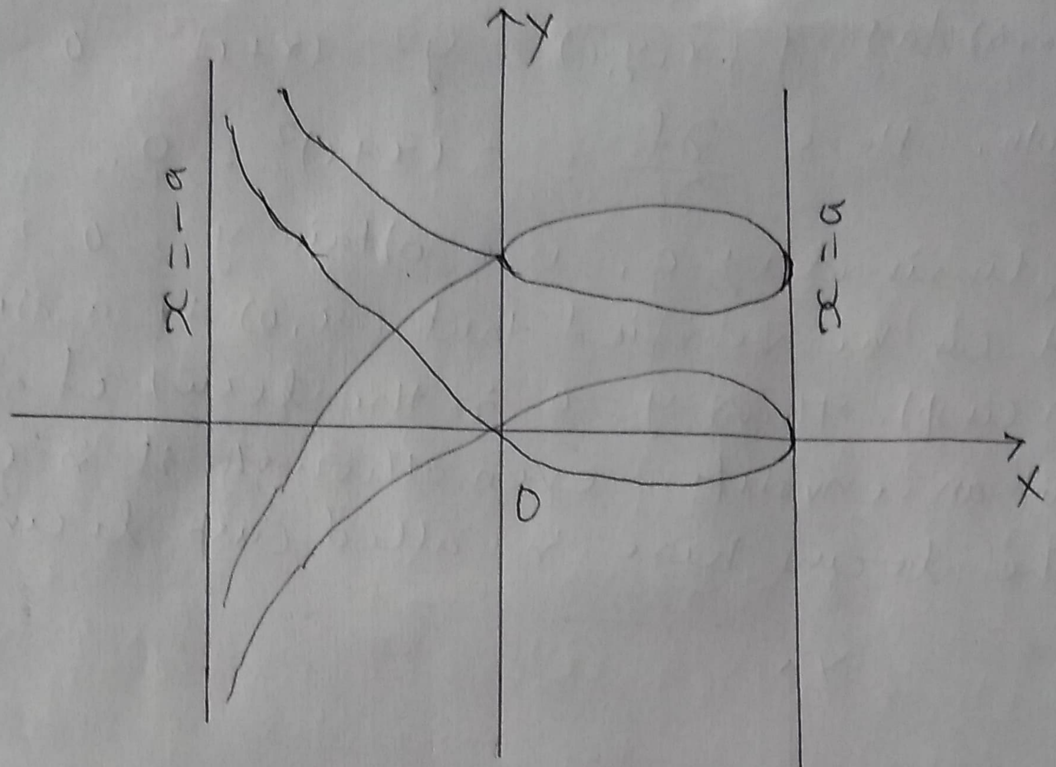
Where a is a constant and c is a parameter.

Soln: Differentiating w.r.t. c, we get

$-2(x+a)(y-c) = 0$.

Eliminating c, we get $x^2(x-a) = 0$.

Thus the c-discriminant consists of two lines $x = 0$ and $x = a$.




If we trace given curve, $(y-c)^2 = \frac{x^2(x-a)}{x+a}$ we shall find that $x=0$ ($y$-axis) is the locus of its singular points (cusp locus) and $x=a$ is the tangent to each curve and hence envelope to the given system of curves.

Further at the points, where $p$'s are equal will include the envelope. (ultimate intersection of consecutive curves, the $p$'s $(= \frac{dy}{dx})$ for the intersecting curves become equal and hence the locus of the points)

Hence the envelope is also contained in the p-discriminant.

Hence, the p-discriminant of $f(x,y,p)=0$ contains the equation to the envelope of $\phi(x,y,c)=0$, which is the solution of the given differential equation.

Thus the envelope is contained both in the c-discriminant as well as in p-discriminant.